International Journal of Theoretical Physics, Vol. 46, No. 8, August 2007 (© 2007)
DOI: 10.1007/s10773-006-9323-9

Reversibility and Regularity

Karl Gustafson[1]

Received August 20, 2006; accepted November 14, 2006
Published Online: February 28, 2007

The Zeno problem of quantum mechanical measurement theory is revisited. A fundamental underlying domain issue is clarified. An alternative formulation for the Zeno problem is given. A new operator-theoretic characterization of reversibility in terms of domain regularity preservation is announced. From these considerations one arrives at a new perspective in which von Neumann's Projection theory and the later Effects theory of Ludwig are seen within an enlarged theory of Measurors and Preparors. It is a Schrödinger picture in which one must be able to account for all wave functions upon which the Hamiltonian can act before one is entitled to draw conclusions about the evolving probabilities.

KEY WORDS: Zeno problem; reversibility; regularity; quantum mechanical measurement.

1. INTRODUCTION

Reversibility in quantum mechanics has assumed renewed importance with the recent advent of the possibilities of quantum computing and more generally the recent advances in quantum information theory and its applications. See for example the book (Nielsen and Chuang, 2000), the proceedings (Antoniou *et al.*, 2003), and much recent literature. One particular issue has been how to control or prevent quantum state decoherence, see e.g. Giulini *et al.* (1996) and Namiki *et al.* (1997), among many others. One proposed way to accomplish that is to use the 'Zeno' effect (Chiu *et al.*, 1977; Misra and Sudarshan, 1977). See for example the proceedings mentioned above for further related recent results and literature citations.

This author was involved in the early formulation of the quantum Zeno effect (Gustafson, 1974, 1975, 1983). However, we were prevented a fully rigorous treatment because of some unresolved operator-theoretic questions. Recently this author returned to these issues (Gustafson, 2003a,b). Beyond these, some new clarity has been achieved. The first goal of this paper is to come up-to-date on

[1] Department of Mathematics, University of Colorado, Boulder, CO 80309-0395, USA; e-mail: gustafs@evclid.colorado.edu.

these results, which have not heretofore been published in the scientific journal venues. In particular, an important unbounded operator domain question central to the Zeno theory is clarified.

The second goal of this paper is to present an alternative formulation of the Zeno problem, in which more attention is given to the state preparation and the state measuring operations. This approach utilizes new mathematical results (Gustafson, 2000) for operator product duals and allows one to remain in Schrödinger picture. This we prefer, in keeping with the belief that the domains $D(H)$ of the Hamiltonians have something to say about the dynamics that one wishes to understand and that if one wishes to avoid these technical considerations by going to the density matrices picture, one has lost some information and also one has perhaps obscured some important issues. The resulting perspective enlarges that of von Neumann's collapse postulate and Ludwig's Effects theory.

The third goal of this paper is to publish a very interesting newly-found operator-theoretic characterization of quantum mechanical reversibility in terms of what will be called here domain regularity preservation. To be more specific, the regularity is that of the totality of the domain $D(T)$ of the infinitesimal generator T of a unitary group U_t or a contraction semigroup Z_t. In particular, any unitary quantum mechanical evolution $U_t = e^{iHt}$ must map $D(H)$ one to one onto itself at every instant. This is a Schrödinger picture, and the result states that the evolution must at all times be able to simultaneously account for all wave functions upon which the Hamiltonian can act before one is entitled to draw conclusions about the overall evolving probabilities.

Section 2 revisits the Zeno problem of quantum mechanical measurement theory. By operator domain-theoretic considerations we clarify when a projected evolution will retain the unitarity property. The central issue here is the denseness of a domain $D(HP)$. Section 3 provides a Zeno alternative in which von Neumann's projections postulate and Ludwig's effects postulate are superseded by what we call here measuror and preparor postulates. Key to this are new results for duals $(AB)^*$ of operator compositions. Section 4 presents a new theorem which shows that the reversibility represented by a quantum mechanical evolution requires that U_t map its generator's domain $D(H)$ one to one onto itself. We call this domain regularity. Section 5 contains concluding remarks.

2. ZENO REVISITED

Our attention was drawn to what became known as the Zeno Paradox by an important early paper of Friedman (1972). In this paper a number if issues concerning quantum mechanical measurement theory were investigated mathematically. In particular, if $P(t)$ is a contraction semigroup on the Hilbert space $\mathcal{H}$, E an orthogonal projection on $\mathcal{H}$, the question of when $s\text{-}\lim_{n\to\infty}(EP(t/n)E)^n$

exists was raised. This would constitute a type of "continuous" observation. Formally one would expect such a limit to be e^{tEAE} where A was the infinitesimal generator of P_t. However, EAE need not in general have the properties of a generator. Note also that the particular instance when $P(t)$ is a unitary evolution $U(t)$ is of major importance and central to a number of quantum measurement issues.

In Friedman (1972) some partial were obtained results via two mathematical approaches: Lie–Trotter product formulas, and quadratic (sesquilinear) forms. However, the main questions remained unanswered. We also did not succeed to answer them (Gustafson, 1974, 1975). When Misra and Sudarshan (1977) published their famous Zeno paper, they also did not answer the underlying fundamental operator-theoretic questions. Instead they abandoned the Schrödinger picture and went to the density matrix picture, where many operator domain questions can be avoided. However, even there it was necessary to assume that the two operator limits $s\text{-}\lim_{n\to\infty} \rho_n(t)$ and $s\text{-}\lim_{n\to\infty} T_n(t)$ exist, where $\rho_n(t) = T_n(t)\rho_0 T_n^*(t)$, where $T_n(t) = (EU(t/n)E)^n$.

We do not wish to give all Zeno history and related mathematical results here. See the cited references. We may mention in particular (Home and Whitaker, 1997; Misra and Antoniou, 2003) but there are many others. It may be asserted here that full mathematical rigor has still not been achieved for many of the important issues. However, the formal physics theory has nonetheless gone ahead, a not unusual situation when one is dealing with quantum measurement theory. A very important experimental result which gave new impetus to Zeno theory was Itano *et al.* (1990).

The following was known early. Let $U(t)$ be an arbitrary unitary evolution with self adjoint infinitesimal generator H and let P be an arbitrary self adjoint bounded projection onto closed subspace M in Hilbert space $\mathcal{H}$. Let $D(T)$ denote the domain of an operator T in $\mathcal{H}$, and $R(T)$ its range. We recall that $PU_t = U_t P$ iff M is a reducing subspace for U_t, and more generally $PU_t P = U_t P$ iff M is an invariant subspace for U_t. More general yet, we have

Lemma 2.1. (Gustafson, 1974, 1975, 1983; Sinha, 1972) *The projected evolution $Z_t = PU_t P$ is a semigroup for all $t \geqq 0$ iff M is a proper subspace without regeneration for U_t, i.e., $PU_s P^{\perp} U_t P = 0$ for all $t, s \geqq 0$.*

We remark that when M is a reducing subspace then Z_t is a unitary group, when M is an invariant subspace then Z_t is a semigroup of partial isometries, and if M fails to meet the requirement of Lemma 2.1, the Z_t loses the semigroup property entirely.

The possibility of retaining a unitary, hence reversible, evolution on so called Zeno subspaces has been investigated in a number of recent books/papers (Facchi *et al.*, 2000, 2001; Namiki *et al.*, 1997; Tasaki *et al.*, 2004) among others. For a better understanding of what is involved in such formulations, we wish to take note of certain unbounded operator-theoretic considerations. The following was clear early.

Lemma 2.2. (Gustafson, 1974, 1975) *Assume $D(HP)$ is dense. Then PHP is symmetric in $\mathcal{H}$, and PHP is selfadjoint if PH is closed in $\mathcal{H}$.*

Proof: Since $D(HP) \equiv D(PHP)$ is dense, $(HP)^*$ and $(PHP)^*$ exist and $(PHP)^* = (HP)^*P \supset (PH)P = PHP$ so PHP is symmetric. If PH is closed, then $PH = (PH)^{**} = (HP)^*$, so one obtains equality in the previous sentence. □

Before looking more closely at the fundamental underlying issue of $D(HP)$ dense, we wish at this point to recall a few operator-theoretic facts which we will use in the remainder of this paper. For more details see the books (Kato, 1980; Riesz and Sz-Nagy, 1955; Weidmann, 1980) among others. In particular, the notion of (closed) invariant subspace M for an unbounded operator T is better thought of in the more general context as a decomposition of T by a direct sum $\mathcal{H} = M \oplus N$ of a pair of subspaces with the requirements that the projection P on M map $D(T)$ into $D(T)$, T maps M into M, T maps N into N. Here P is the (generally oblique) projection of M along N. Such decomposition of T is equivalent to T commuting with $P : PT \subset TP$. Then $TP = PTP = PT$ on $D(T)$. When T is a selfadjoint operator H and P an orthogonal projection and $U_t = e^{iHt}$, then when one says that M reduces H one is saying all of the following: $PH \subset HP$, $P : D(H)$ into $D(H)$, $(H - zI)^{-1}P = P(H - zI)^{-1}$ for all z with $Im z \neq 0$, $PE(s) = E(s)P$ for all real s and all spectral family projectors of H, and $PU_t = U_tP$ for all real t. Our point-of-view is that such a simplified situation is not that of the Zeno issues.

A second set of facts to remember is that T^* exists iff $D(T)$ is dense, that $D(T^*)$ need not be dense, but $D(T^*)$ is dense iff T is closable, and then its closure satisfies $\overline{T} = T^{**}$. For any two densely defined operators A and B and if AB is densely defined, then $(AB)^* \supset B^*A^*$, with equality if $A \in \mathcal{B}(\mathcal{H})$. Other conditions for equality will be described later. We also will use other unbounded operator theory, such as the associativity $T_1(T_2T_3) = (T_1T_2)T_3$, without comment.

We may now sharpen Lemma 2.2 to further clarify the situation.

Theorem 2.1. *$D(HP)$ is dense iff PH is closable. Then $(HP)^*$ is defined and $(HP)^* = \overline{PH} \supset PH$ has domain at least as large as $D(H)$. Thus generally $(PHP)^* = \overline{PH}P$ whenever PH is closable. Furthermore the polar factors satisfy $|\overline{PH}| = (HP^2H)^{1/2}$ and $|HP| \supset (PH^2P)^{1/2}$.*

Proof: Because PH is densely defined, its adjoint exists and is $(PH)^* = HP$. This operator is densely defined iff PH is a closable operator. Then HP is a closed densely defined operator and $(HP)^* = (PH)^{**} = \overline{PH} \supset PH$. To obtain the polar factors we form $(\overline{PH})^* PH = HP^2 H$ and $(HP)^* HP = \overline{PH} HP \supset PH^2 P$. $\square$

We mention that Theorem 2.1 holds for arbitrary selfadjoint operator H and arbitrary orthogonal projection P. Therefore one cannot expect to get much more from it.

The key assumption that HP be densely defined is made throughout the recent treatment (Exner and Ichinose, 2003) of quantum Zeno dynamics. Their approach of quadratic forms follows that of Friedman (1972) and one is concerned with the form $||H^{1/2} Pu||^2$ with form domain $D(H^{1/2} P)$. The operator $H_P := (H^{1/2} P)^*(H^{1/2} P)$ is associated with this form. It is noted there that H_P may not be densely defined but that then it will be a selfadjoint operator in some closed subspace of $\mathcal{H}$. From our analysis here we would like to defer slightly, or at least point out some ambiguity in such a conclusion. When $H^{1/2} P$ is not densely defined in $\mathcal{H}$, one cannot even speak of an operator $(H^{1/2} P)^*(H^{1/2} P)$. Of course one can then reduce one's considerations to the smaller Hilbert space M. But if $H^{1/2} P$ was not densely defined in $\mathcal{H}$, then $H^{1/2}$ will not be densely defined in M either. Moreover, whatever its domain there, the range $R(H^{1/2}|_M)$ will generally fall at least partially outside of M.

The same reservation applies to the analysis of Facchi and Pascazio (2003) The formulation there combines continuous measurement with a coupling limit to force the system to evolve in a set of orthogonal subspaces of the parent Hilbert space. These quantum Zeno subspaces are the eigenspaces of a Hamiltonian which is supposed to represent the interaction between the evolving quantum dynamical system and the measurement apparatus. The use of a superselection rule and an adiabatic theorem are assumed to determine "the subspaces that the apparatus is able to distinguish." Thus the physical description is now that of a dynamical evolution allowing changing Zeno subspaces. However, from our point of view, since the modeling and analysis is carried out in the density matrix formulation, its underlying rigorous validity, e.g., the denseness of the domains of the effective Hamiltonians in the individual Zeno subspace evolutions, has not been considered. A similar comment applies to the von Neumann subalgebras Zeno formulation (Schmidt, 2002), especially if one does not want the measuring projection E to have to be within the Hamiltonian's functional calculus.

A number of conditions are known which can render PH closable, hence $D(HP)$ dense. For example, since PH is densely defined, it is sufficient that its numerical range $W(PH)$, the set of all inner product values $\langle PHx, x\rangle$ over all x in $D(H)$ with $||x|| = 1$, not be the whole complex plane, e.g. see Gustafson and Rao (1997). However, closability issues can be both delicate and stubborn.

3. A ZENO ALTERNATIVE

From entirely different motivations, this author obtained the following results for the general question of when $(AB)^* = B^*A^*$ for (generally) unbounded densely defined operators A and B in a Hilbert space $\mathcal{H}$.

Lemma 3.1. Gustafson (2000) *Let A and B be arbitrary densely defined operators and suppose $D(AB)$ is dense, $R(B) \supset D(A)$, $D(B^*) \supset R(A^*)$, and B is 1-1. Then $(AB)^* = B^*A^*$. In particular when A and B are selfadjoint, then the conditions are $D(AB)$ dense, $R(B) \supset D(A)$, $D(B) \supset R(A)$.*

From these considerations one obtains the following.

Theorem 3.1. (Gustafson, 2003a,b) *Let A be a "continual measurement observable" $A = A^*$ bounded, $R(A) \supset D(H)$, $D(HA)$ dense. Then AHA is selfadjoint and the exponentiation e^{iAHAt} is a unitary evolution.*

Theorem 3.2. (Gustafson, 2003a,b) *Let A, B, C be densely defined operators in a Hilbert space $\mathcal{H}$. Suppose the domains $D(BC)$ and $D(ABC)$ are dense, ranges $R(BC) \supset D(A)$, $R(C) = D(B)$, domains $D((BC)^*) \supset R(A^*)$, $D(C^*) \supset R(B^*)$, and C and BC are 1-1. Then $(ABC)^* = C^*B^*A^*$.*

Corollary 3.1. (Gustafson, 2003a,b) *Let A and H be general selfadjoint operators and suppose $D(HA)$ and $D(AHA)$ are dense, $R(HA) \supset D(A)$, $R(A) \supset D(H)$, $D((HA)^*) \supset R(A)$, $D(A) \supset R(H)$. Then AHA is selfadjoint and e^{iAHAt} is a unitary evolution.*

A few remarks here. First, the old Fredholm theory always gave $(AB)^* = B^*A^*$ when A and B were densely defined (generally unbounded) Fredholm operators. Some slightly more general results also held, see Gustafson (1969). But when A is a projection P in the Zeno context, we do not believe that it is natural to have to assume a Fredholm finite index condition for P.

From the above results we may now formulate a Zeno Alternative, the goal being decoherence suppression and reversibility by maintaining a unitary evolution, but with more attention given to the state preparation and measuring operations. The infinitesimal generator AHC is to be prepared by C and measured by A. For example, if we focus attention on AHA one may think of allowing A to be in the Effects class $0 \leqq A \leqq 1$, e.g., see (Davies, 1976; Kraus, 1983), thereby escaping the restriction to the von Neumann Projections hypothesis. However, the formulation here is more general. For example, the condition $R(A) \supset D(H)$ may be interpreted as requiring that the preparor A be able to prepare all envisioned wave functions for the domain of the Hamiltonian H. In like manner the condition

$D(A) \supset R(H)$ can be interpreted as meaning that A as measuror should be able to measure all wave functions after they are operated on by H. When we take both preparor and measuror to be the same operator A as in Corollary 3.1, it can be seen that in fact one has required that $R(H) = D(A)$, an admittedly strict condition, which may be relaxed by allowing different preparors and measurors.

One may regard this Zeno Alternative, which allows unitary evolutions to continue during the processes of state preparation and measurement, as an enlarged picture of the quantum mechanical measurement theory, according to

$$\begin{matrix} \text{Projections} \\ \text{(von Neumann)} \end{matrix} \subset \begin{matrix} \text{Effects} \\ \text{(Ludwig)} \end{matrix} \subset \begin{matrix} \text{Measurors/Preparors} \\ \text{(Gustafson)} \end{matrix} .$$

The earlier formulations are preserved within the larger formulation. The noteworthy property of measurors and preparors is that they must have dense domains and dense ranges in order to account for all possible physically meaningful probabilities. As a result of this, there need not be any "wave function collapse" because one has allowed "complete measurement."

In (Gustafson, 2003a,b) this Zeno alternative is brought to bear on the (Friedman, 1972) Counter model, in which P is implemented by multiplication by the characteristic function $\chi(\mathcal{E})$ where $\mathcal{E}$ represents a closed bounded three-dimensional domain with smooth boundary $\partial\mathcal{E}$. One of the difficulties of Friedman's model is that there are two different Hamiltonians posed, both called H_0. The first is the free space Laplacian Δ in $\mathcal{L}^2(R^3)$. the second is the Laplacian Δ in $\mathcal{L}^2(\mathcal{E})$ with Dirichlet boundary conditions tacitly assumed. The domains of these two Hamiltonians are well-known but quite different. Moreover, we know rather generally that exponentiations U_t commute with their infinitesimal generators, i.e., $U_t H_0 \subseteq H_0 U_t$. This means in particular that U_t maps $D(H_0)$ into $D(H_0)$. So as argued in (Gustafson, 2003a,b), if one starts with a free space evolution $e^{-it\Delta}\psi_0$ and one wants to 'count it' by projecting it to $\mathcal{L}^2(\mathcal{E}) \cap D(\Delta_0)$ where Δ_0 denotes the Laplacian with Dirichlet (trace) boundary condition, thereafter the 'counter evolution' $e^{it\Delta_0}P\psi_0$ becomes a $\mathcal{L}^2(\mathcal{E})$ unitary wave packet which continues to propagate within the counter while maintaining the property of always vanishing on the counter boundary $\partial\mathcal{E}$. Once you are in the counter, you keep evolving there forever. However, if you had imposed any other kind of self-adjoint boundary condition on the surface of the counter, after counting, that boundary condition is also forever retained within the continuing counter evolution. This view differs from that put forth in Facchi *et al.* (2001).

Similar domain-theoretic ambiguities may be found in the physical literature of Zeno subspaces, Zeno control, Decoherence-free unitary evolutions, dynamical decoupling, suppression of system-environment interaction, quantum-computing maintenance of reversibility by other means, etc. This technology is difficult, both in theory and experimental realization, and must go forward. However, the Zeno Alternative we offer here attempts to explain operator-theoretically how to *not* collapse the wave packet in the measuring process, or how to keep re-preparing the wave function to maintain reversibility. From the experimental point of view a similar philosophy was expressed in Pascazio *et al.* (1993).

4. REVERSIBILITY AND REGULARITY

It is well known that the Heat equation semigroup greatly smoothes the initial data. For example, see (Gustafson, 1999, pp. 128–131), solutions $u(x, t)$ to the Heat equation $u_t - u_{xx} = 0$, $-\infty < x < \infty$, $t > 0$, $u(x, 0) = f(x)$, $-\infty < x < \infty$, for any initial value f in any $\mathcal{L}^P$ class, become C^∞ immediately for $t > 0$. And the Heat equation is quite irreversible, in the sense that the backward heat equation is unstable. On the other hand, the reversibility of a quantum mechanical Schrödinger evolution corresponds just to the unitarity of its semigroup e^{itH}. The following result grew out of such thoughts combined with the considerations about domains elsewhere in this paper.

Theorem 4.1. (Reversibility $\Rightarrow$ Regularity). *A unitary group U_t necessarily exactly preserves its infinitesimal generator's domain $D(H)$. That is, U_t maps $D(H)$ one to one onto itself, for all $-\infty < t < \infty$.*

Proof: For every $-\infty < t < \infty$ the linear isometry $||U_t x|| = ||x||$ property and the commutativity property $U_t H \subset H U_t$ guarantee that U_t maps $D(H)$ one to one into $D(H)$. Is it onto $D(H)$? Suppose not. Then for some t there exists an x in $D(H)$ which is not in the image $U_t(D(H))$. Apply U_t^{-1} to x. By the commutativity property again, we know $z = U_{-t}x$ is in $D(H)$. But then $U_t z = x$ must have been in $D(H)$. □

Let us briefly discuss some facets of this Theorem.

First, the result is quite evident and natural once one sees it. Therefore it may exist elsewhere in the literature, but in a limited search, we did not find it. It importantly distinguishes the special action of U_t on $D(H)$ from its one to one onto action on the whole Hilbert space.

Second, one can prove it other ways. For example, just from the operator state diagram theory (Gustafson, 1997) one knows immediately that the 7 possible operator-adjoint state combinations for densely defined closed operators in a Hilbert space are reduced to two for U_t, namely, states I_1 and III_1, just by the fact that U_t has a bounded inverse. Then the 1-1ness of U_t^* places you in combined state $I_1 I_1$. However, this tells you nothing about the specific action of U_t on $D(H)$. Still, rather than abandoning this alternate proof approach, we wish to push it through for reasons that will become apparent below, and also because we have not seen these considerations elsewhere.

Lemma 4.1. *Let T be a closed densely defined operator in a Hilbert space $\mathcal{H}$ and let $U_t = e^{itH}$ be a unitary evolution there which commutes with T. Then U_t remains unitary on the graph-norm Hilbert space $\mathcal{H}_T$.*

Proof: $\mathcal{H}_T$ is the Hilbert space $D(T)$ equipped with the inner product $\langle x, y\rangle_T = \langle x, y\rangle + \langle Tx, Ty\rangle$. By the commutativity $U_t T \subseteq TU_t$ we have the isometry property retained,

$$||U_t x||_T^2 = ||U_t x||^2 + ||TU_t x||^2 = ||x||^2 + ||U_t Tx||^2 = ||x||_T^2$$

Also the adjoint U_t^* considered in $\mathcal{H}_T$ is the same as the original adjoint, using again the commutativity,

$$\begin{aligned} \langle U_t x, y\rangle_T &= \langle U_t x, y\rangle + \langle U_t Tx, Ty\rangle \\ &= \langle x, U_t^* y\rangle + \langle Tx, TU_t^* y\rangle = \langle x, U_t^* y\rangle_T \end{aligned}$$

Thus $U_T^* = U_t^{-1}$ in the original space carries over to $\mathcal{H}_T$. □

In particular, Lemma 4.1 provides an alternate proof of Theorem 4.1. Let T be H. Then by the state diagram argument given above, U_t and U_t^* are both onto $\mathcal{H}_T = D(H)$.

Lemma 4.2. *Under the conditions of Lemma 4.1, a contraction semigroup evolution $Z_t = e^{tA}$ remains a contraction semigroup on $\mathcal{H}_T$.*

Proof: As above. Note that the semigroup commutativity property $Z_t T \subseteq TZ_t$ is essential, as is its sub-property that Z_t map $D(T)$ into $D(T)$. □

Now we can provide a partial converse to Theorem 4.1 in the sense of asking, suppose a contraction semigroup Z_t exhibits the regularity preservation property that it map the domain $D(A)$ of its infinitesimal generator one to one onto itself. What 'unitarity' properties does Z_t exhibit for all $t \geq 0$?

Proposition 4.1. *Let $Z_t = e^{tA}$ be a contraction semigroup on a Hilbert space $\mathcal{H}$ such that Z_t maps $D(A)$ one to one onto $D(A)$. Then Z_t on $\mathcal{H}_A$ can be extended to a group $Z_{-t} = Z_t^{-1}$.*

Proof: By Lemma 4.2 above with $A = T$, we know Z_t remains a contraction semigroup on $\mathcal{H}_A$. Because Z_t maps $\mathcal{H}_A$ 1-1 onto itself, we know (e.g., see Gustafson, 1997), Z_t^{-1} is bounded and also maps $\mathcal{H}_A$ 1-1 onto itself. By known results in semigroup theory (e.g., see Riesz and Sz-Nagy, 1955, p. 393, we may extend Z_t to a group by defining $Z_{-t} = Z_t^{-1}$). □

We mention that Z_t^* remains the same in $\mathcal{H}_A$ as it was in $\mathcal{H}$, so that one can relate the adjoint semigroups by $(Z_t^*)^{-1} = (Z_t^{-1})^*$, pursue further converse statements for the original Hilbert space $\mathcal{H}$, etc., which we will not do here. One reason we present these results here is in hopes that they be used elsewhere by those working in Zeno theory. To that end we add a few more comments here. Note that in Lemma 4.1 you cannot conclude that T remain a closed operator in H_T unless T maps $D(T)$ into itself; and then T becomes a bounded operator. It was because we did not wish to make such assumptions that we did not publish (Gustafson, 1974, 1975). Note that the heat equation converts its infinitesimal generator's domain $W^{2,2}(-\infty, \infty)$ immediately into C^∞ functions, so in a sense it is an extreme opposite to Theorem 4.1. It would be interesting to know more about semigroup behavior in-between Theorem 4.1 and Proposition 4.1. For example, we know a semigroup Z_t of isometries has a unitary group extension on a larger Hilbert space.

Third, elsewhere (Gustafson, 1997) this author has developed a somewhat related principal of regularization, which postulates regularity increase in Nature as a very extensive Second Law. A number of examples were given to show that Nature generally prefers regularizing processes. Entropy increase sometimes coincides with this regularity preference. Thus, Time-asymmetric physics develops as Nature smoothes data. Coupled with such macroscopic regularization one often finds a corresponding microscopic refinement of detail.

What Theorem 4.1 states within such a context is that in unitary quantum mechanical evolutions, one cannot lose any of the totality of detail embodied in the totality of wave functions ψ in $D(H)$. It is quite interesting to think about how much 'mixing around' of $D(H)$ the evolution U_t can do. Perhaps one will investigate such questions elsewhere, e.g., from the viewpoint of ergodic theory. However, just by itself, Theorem 4.1 says that to have reversibility in quantum mechanics, you cannot lose a single wave function from $D(H)$ as you proceed forward in time. In other words, you must be able to account for all evolving probabilities, future and past, all of the time.

This finding also gives renewed importance to the role of preparation of states in quantum mechanics. The ensemble $\{\psi_0\}$ of experimental initial states which are prepared to simulate *a posteriori* in an experiment what you could expect *a priori* from all possibilities from $D(H)$, must be sufficiently extensive within $D(H)$ and in such a way that their trajectories continue to predict properly during evolution. Ideally, you need a dense subset from $D(H)$.

To emphasize the above discussion, which to us seems important, we may state

Corollary 4.1. *A quantum mechanical evolution U_t must and does continuously and simultaneously account for all probabilities $|\psi(t)|^2$ and all wave functions $\psi(t)$ in the domain of its Hamiltonian.*

It is important to note that Theorem 4.1 is of wider interest, i.e., it will apply to any unitary evolution $U_t = e^{itH}$ in any context. For example, even if one has "resorted" to the Heisenberg interaction picture and to unitary evolutions $\rho(t) = U_t \rho_0 U_t^*$, one knows from Theorem 4.1 that the underlying Hamiltonian must have the domain regularity preservation property. Perhaps more interestingly, effective "interaction Hamiltonians" such as those assumed in quantum dynamical decoupling (Tasaki *et al.*, 2004; Viola *et al.*, 1999) must also enjoy and respect the domain regularity preservation property. Our point of view is that in such situations, this is a physical property that one should try to understand mathematically, in order to obtain a deeper understanding of the physics. As is well-known, selfadjointness is a powerful property and often depends on, and represents, the correct physical boundary conditions.

5. CONCLUDING REMARKS

We have stayed within the Schrödinger description of quantum dynamics for three reasons. First, that is where we began, when first looking at these issues 30 years ago. Second, in our opinion, that is where the dynamics takes place. The difficulties of quantum measurement theory do not impugn in any way the validity of the Schrödinger partial differential equations which describe atomic and molecular dynamics. Rather, the difficulties and paradoxes of quantum measurement theory are self-induced, as are most paradoxes, by some defect in how we are modeling the measurement problem. Putting it perhaps a bit too bluntly, we do not understand well enough the microphysical nature of the physical entities we want to measure. So we do the best we can, by speaking of particles, wave functions, probabilities, expected values. Third, there is some very beautiful mathematical operator theory which is unavoidably intrinsic to the physical descriptions within which we are trying to work. The difficulties of this operator theory mathematics seem to match and indicate the corresponding inadequacies in our understandings of the physics.

Von Neumann's original projection postulate had three advantages. First, he wished to have a model within a Hilbert space context. Of course we know now that aspects of quantum mechanics, such as improper wave functions over the continuous spectrum, resonances, field theories, take us beyond the Hilbert space context. But it is a good context. Second, modeling a measurement by projection onto an eigenspace agreed very well with physical experiment. The spectral lines of Hydrogen, Helium, and the other elements, as found by optical spectroscopy in the experimental laboratories, agree brilliantly with those predicted by the eigenfunctions of the Schrödinger partial differential equation. Of course, those projections were within the Hamiltonian's spectral calculus, and hence did not deal at all with the actual process of measurement. Third, Hilbert space, e.g. the $\mathcal{L}^2(\mathbb{R}^n)$ spaces, were a very natural way to accommodate the wave function probability $|\psi(t)|^2$ interpretations of Born. In this paper we have sharpened our understanding of that feature.

We have not attempted any wide review of quantum Zeno dynamics, quantum theory of open systems, quantum computation and control of decoherence, quantum statistical mechanics, others. We have purposely avoided all issues about how our theory might help or conflict with other Zeno investigations based upon the Heisenberg picture, open systems formulations, others. We would, however, like to mention that we have looked at some of those formulations, and we are of the opinion that usually they mask important underlying issues whose fundamental understanding would require domain-theoretic explanation such as we have presented in this paper. There is a lot of future mathematical investigation needed to make rigorous all of those interesting formulations. Finally, it should be clear that our measuror/preparor observable theory is in its infancy, and that to be complete, it will need to be extended to time-dependent measurors and preparors, and to how they couple to the actual measurement and preparation processes, depending on what one wishes to measure/prepare. In particular we want to stress that we envision all of the considerations of this paper, the domain-theoretic results, the Zeno Alternative, to also be applied to subspace evolutions that one wants to be reversible. The measuror/preparor ansatz will then need to be tailored to each of those situations.

ACKNOWLEDGMENTS

The author would like to express his appreciation of Professor Saverio Pascazio, who he first met at the Solvay XXII Congress in Delphi, Greece, in November 2001. Professor Saverio cornered the author, stating that he had heard persistent rumors that the author had been one of the originators of the Zeno problem (true), and he pressed the author to publish the results and understandings known at that time, not only for the history of the problem, but also to perhaps clarify certain present issues. Moreover Professors Saverio and Scardicchio then posted a preliminary version of Gustafson (2003a) to the Los Alamos archives. That paper and the present one are the author's attempt to come somewhat up to date on these physical and mathematical issues which, in our opinion, are far from resolved.

REFERENCES

Antoniou, I., Sadovnichy, V., and Walther, H. (Eds.). (2003). *The Physics of Communication*, World Scientific, Singapore.

Chiu, C., Sudarshan, G., and Misra, B. (1977). Time evolution of unstable quantum states and a resolution to Zeno's paradox. *Physical Review D* **16**, 520–529.

Davies, E. (1976). *Quantum Theory of Open Systems*, Academic Press, New York.

Exner, P. and Ichinose, T. (2003). Product formula related to quantum Zeno dynamics. *Proceedings of the XIV International Congress on Mathematical Physics*, Lisbon, World Scientific (to appear). See also arXiv:math-ph/0302060.

Facchi, P., Gorini, V., Marmo, G., Pascazio, S., and Sudarshan, G. (2000). Quantum Zeno dynamics. *Physics Letters A* **274**, 12–19.

Facchi, P., Pascazio, S., Scardicchio, A., and Schulman, L. (2001). Quantum dynamics yields ordinary constraints. *Physical Review A* **65**, 012108.

Facchi, P. and Pascazio, S. (2003). Quantum Zeno subspaces and dynamical superselection rules. In Antoniou, I., Sadovnichy, V., and Walther, H. (Eds.), *The Physics of Communication*, World Scientific, pp. 251–286.

Friedman, C. (1972). Semigroup product formulas, compressions, and continual observations in quantum mechanics. *Indiana University Mathematics Journal* **21**, 1001–1011.

Giulini, D., Joos, E., Kiefer, C., Kupsch, J., Stamatescu, I., and Zeh, H. (1996). *Decoherence and the Appearance of a Classical World in Quantum Theory*, Springer, Berlin.

Gustafson, K. (1969). On projections of selfadjoint operators and operator product adjoints. *Bulletin American Mathematical Society* **75**, 739–741.

Gustafson, K. (1974). On the "Counter Problem" of quantum mechanics. *Unpublished*, pp. 14.

Gustafson, K. (1975). Some open operator theory problems in quantum mechanics. Rocky Mountain Mathematics Consortium Summer School on C^* Algebras, Bozeman, Montana. Unpublished notes, pp. 7.

Gustafson, K. (1983). Irreversibility questions in chemistry, quantum-counting, and time-delay. In Hinze, J. (Ed.), *Energy Storage and Redistribution in Molecules*, Plenum Press, pp. 516–526.

Gustafson, K. (1997). Operator spectral states. *Computers Applications of Mathematics* **34**, 467–508.

Gustafson, K. (1999). *Partial Differential Equations and Hilbert Space Methods*, Dover Publications, Mineola, N.Y.

Gustafson, K. (2000). A composition adjoint lemma. In Gesztesy, F., Holden, H., Jost, J., Paycha, S., Röckner, M., and Scarlatti, S. (Eds.), *Stochastic Processes, Physics, and Geometry: New Interplays*, Vol. II, American Mathematical Society, pp. 253–258.

Gustafson, K. (2003a). A Zeno story. *Quantum Computers and Computing (Moscow)* **35**(2), 35–55. See also quant-ph/0203032.

Gustafson, K. (2003b). The quantum Zeno paradox and the Counter problem. In Khrennikov, A. (Ed.), *Foundations of Probability and Physics-2*, Växjo University Press, Sweden, pp. 225–236.

Gustafson, K. and Rao, D. (1997). *Numerical Range: The Field of Values of Linear Operators and Matrices*, Springer, Berlin.

Home, D. and Whitaker, M. (1997). A conceptual analysis of quantum Zeno; paradox, measurement, and experiment. *Annals of Physics* **238**, 237–285.

Itano, W., Heinzen, D., Bollinger, J., and Wineland, D. (1990). Quantum Zeno effect. *Physical Review A* **41**, 2295–2300.

Kato, T. (1980). *Perturbation Theory for Linear Operators*, Springer, Berlin.

Kraus, K. (1983). *States, Effects, and Operations*, Springer Lecture Notes in Physics 190, Springer, Berlin.

Misra, B. and Sudarshan, G. (1977). The Zeno's paradox in quantum theory. *Journal of Mathematical Physics* **18**, 756–763.

Misra, B. and Antoniou, I. (2003). Quantum Zeno effect. In Antoniou, I., Sadovnichy, V., and Walther, H. (Eds.), *The Physics of Communication*, World Scientific, pp. 233–250.

Namiki, M., Pascazio, S., and Nakazato, H. (1997). *Decoherence and Quantum Measurements*, World Scientific, Singapore.

Nielsen, M. and Chuang, I. (2000). *Quantum Computation and Quantum Information*, Cambridge University Press, U.K.

Pascazio, S., Namiki, M., Badurek, G., and Rauch, H. (1993). Quantum Zeno effect with neutron Spin. *Physics Letters A* **179**, 155–160.

Riesz, F. and Sz-Nagy, B. (1955). *Functional Analysis*, Ungar Publishing, New York.

Sinha, K. (1972). On the decay of an unstable particle. *Helvetica Physica Acta* **45**, 621–628.

Schmidt, A. (2002). Zeno dynamics of von Neumann algebras. *Journal of Physics A* **35**, 7817–7825.
Tasaki, S., Tokuse, A., Facchi, P., and Pascazio, S. (2004). Control of decoherence: Dynamical decoupling versus quantum Zeno effect. A case study for trapped ions. *International Journal of Quantum Chemistry* **98**, 160–172.
Viola, L., Knill, E., and Lloyd, S. (1999). Dynamical decoupling of open quantum systems. *Physical Review Letters* **82**, 2417–2421.
Weidmann, J. (1980) *Linear Operators in Hilbert Spaces*, Springer, New York.